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What Could Be Worse than the Butterfly Effect?

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Our understanding of classical mechanics (CM) has undergone significant growth in the latter half of the twentieth century and in the beginning of the twenty-first. This growth has much to do with the explosion of interest in the study of nonlinear systems in contrast with the focus on linear systems that had colored much work in CM from its inception. For example, although Maxwell and Poincaré arguably were some of the first to think about chaotic behavior, the modern study of chaotic dynamics traces its beginning to the pioneering work of Edward Lorenz (1963). This work has yielded a rich variety of behavior in relatively simple classical models that was previously unsuspected by the vast majority of the physics community (see Hilborn 2001). Chaos is a property of nonlinear systems that is usually characterized by sensitive dependence on initial conditions (SDIC). In CM the behavior of simple physical systems is described using models (such as the harmonic oscillator) that capture the main features of the systems in question (Giere 1988). Given two identical linear models of CM in nearly identical initial states, they will evolve in much the same way over long time scales. Two identical nonlinear models exhibiting SDIC in nearly identical states, however, will evolve in radically different ways in a relatively short time period because of extreme sensitivity to the smallest changes in initial states.

This latter behavior has been popularized in the provocative 'butterfly effect,' the idea that the flapping of a butterfly's wings in Argentina could cause a tornado in Texas three weeks later, say. An even more provocative version of such an effect is the possibility that a quantum fluctuation in Argentina could cause a tornado in Texas five weeks later! Jesse Hobbs (1991) and Stephen Kellert (1993) have argued that quantum events can influence classical systems through SDIC. These sensitive dependence (SD) arguments raise interesting questions about whether CM is indeterministic (Bishop and Kronz 1999) and have tantalizing implications for larger philosophical issues such as free will/determinism debates (e.g., Kane 1996; Bishop 2002a).

This discovery of SDIC in nonlinear models runs counter to the textbook vision of CM, a vision guided by an almost exclusive focus on linear systems. Therefore, it is important to clearly distinguish between linear and nonlinear systems along with establishing some basic terminology (§I). The notions of SDIC and chaos also need clarification, since they play crucial roles in SD arguments. This will require some discussion of Lyapunov exponents as well as the relationship between nonlinear dynamics and chaos (§II and Appendix). For the sake of concreteness, it will also be useful to focus on the Laplacean vision for classical particle mechanics (e.g., Bishop 2002b, 2003 and 2005a), particularly the crucial notion of unique evolution (§III). The SD argument can then be stated in a clear form and its defenses, criticisms and limitations assessed in the more general context of nonlinear dynamics (§IV). Concluding remarks follow (§V).

Before jumping in, there is a fundamental question here about whether a CM model of a pendulum is really just an approximation of a full quantum treatment. This is a very deep issue, but one reason to think that classical and quantum systems should be treated as distinct is that the relationship between quantum mechanics (QM) and CM is much subtler than either an approximation treatment or correspondence principles suggest (Primas 1998; Bishop 2005b). There is no space here to treat this question, but the following analysis will presuppose the distinctness of classical and quantum systems.

I Linearity and Nonlinearity

Theoretical models in physics include equations of motion (often referred to as dynamical or evolution equations) describing the change in time of variables taken to adequately describe some physical system of interest. In addition, a complete specification of the initial state referred to as the *initial conditions* (ICs) for the model and/or a characterization of the boundaries for the model domain known as the *boundary* *conditions* (BCs) are also required. As a simple example of a CM model, suppose we wanted to study the firing of a rubber ball at a wall by a small cannon. The BC might be that the wall absorbs no kinetic energy (energy of motion) so that the ball is reflected off the wall with no loss of energy. The ICs would be the initial position and velocity of the ball as it left the mouth of the cannon. The equation of motion would then describe the flight of the ball to and from the wall.

1. State Space

Much of the analysis of physical systems takes place in what is called *state space*, an abstract mathematical space of points where, each point represents a possible state of the system. An instantaneous state is taken to be characterized by the instantaneous values of the variables considered crucial for a complete description of the state. One advantage of working in state space is that it often allows us to study the geometric properties of the trajectories of the system in question without knowing the exact solutions to the evolution equations. When the state of the system is fully characterized by position and momentum variables, the resulting space is often called *phase space*.[1] In typical dynamical models, the coordinates of such a space are the generalized momenta and positions.[2] A model can be studied in state space by following its trajectory from the initial state (q_o, p_o) to some chosen final state (q_f, p_f) (Figure 1). The evolution equations govern the path — the history of state transitions — of the system in state space.

However, note that there are crucial assumptions being made here. Namely, that a state of a system is characterized by the values of the crucial variables and that a physical state corresponds via these values to a point in state space. These assumptions allow us to develop mathematical models for the evolution of trajectories in state space and such models are taken to represent (perhaps through an isomorphism or some more complicated relation) the physical systems of interest. In other words, we assume that our mathematical models are faithful representations of physical systems and that the state spaces employed faithfully represent the space of physical possibilities of target systems. This package of assumptions is called the *faithful model assumption*, and,

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1 The differences between state space (general case) and phase space (special case) are often ignored in the literature.

2 These generalized coordinates allow for systems to be characterized by variables other than linear momentum and position (e.g., angles and angular momentum).

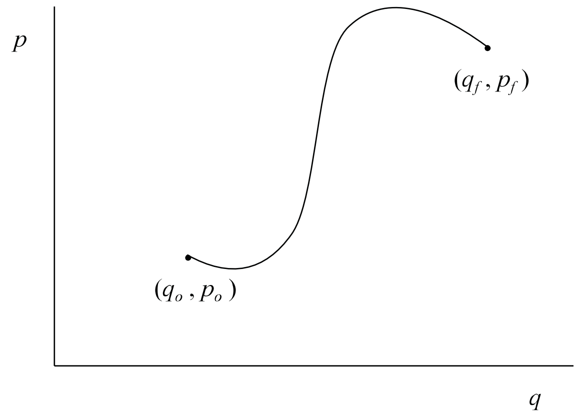

Figure 1.
Path of a physical system in phase space.

in its idealized limit — *the perfect model scenario* — it can license the (perhaps sloppy) slide between model talk and system talk. Although I doubt the perfect model scenario can be defended in nonlinear contexts, I will grant this assumption for the sake of discussion as the argument I wish to examine is formulated within this scenario.

2. Linear and Nonlinear Systems

A dynamical system is characterized as linear or nonlinear depending on the nature of the equations of motion describing the target system. For concreteness, consider a differential equation system, such as $d\boldsymbol{x}/dt = \boldsymbol{F}\boldsymbol{x}$ for a set of variables $\boldsymbol{x} = x_1, x_2, \ldots, x_n$. These variables might represent positions, momenta and other key features of the target system, and the system of equations tells us how these key variables change with time. The system of equations is linear if the matrix of coefficients $\boldsymbol{F}$ does not contain any of the variables $\boldsymbol{x}$ or functions defined in terms of them (e.g., sin x), though functions of time are allowed; otherwise it is nonlinear. For example, consider the nonlinear Lorenz model equations for convection in fluids (Lorenz 1963):

$$\begin{aligned}\frac{dx}{dt} &= -\sigma x + \sigma y \\ \frac{dy}{dt} &= -xz + rx - y \\ \frac{dz}{dt} &= xy + bz\end{aligned} \quad . \qquad \textbf{(1)}$$

Here $\boldsymbol{F}$ has the Form

$$\begin{pmatrix} -\sigma & \sigma & 0 \\ -z+r & -1 & 0 \\ 0 & x & b \end{pmatrix}, \qquad \textbf{(2)}$$

where the variables z and x appear as coefficients, rendering the system of equations nonlinear.

Suppose that $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$ are solutions of the equation system $d\boldsymbol{x}/dt = \boldsymbol{F}\boldsymbol{x}$. If the system of equations is linear, it can easily be shown that $\boldsymbol{x}_3(t) = a\boldsymbol{x}_1(t) + b\boldsymbol{x}_2(t)$ is also a solution, where a and b are constants. This is known as the *principle of linear superposition*. So if the matrix of coefficients $\boldsymbol{F}$ does not contain any of the variables $\boldsymbol{x}$ or functions of them, then the principle of linear superposition holds. If the principle of linear superposition holds, then, roughly, a system behaves linearly if any multiplicative change in a variable, by a factor a say, implies a multiplicative or proportional change of its output by a. For example if you start with your stereo at low volume and turn the volume control one unit, the volume increases one unit. If you now turn the control two units, the volume increases two units. These are examples of *linear responses*. In a nonlinear system, such as (1), linear superposition does not hold and a system need not change proportionally to the change in a variable. If you turn your volume control too far, the volume not only increases more than the number of units of the turn, but whistles and various other distortions occur in the sound. These are examples of *nonlinear responses*.

II Sensitive Dependence, Lyapunov Exponents and Chaos

1. Defining SDIC and Lyapunov Exponents

Although often discussed in the context of chaos, SDIC is more generally a phenomenon of nonlinear systems and models given appropriate values of key parameters (e.g., σ, b, and r in the case of the Lorenz model). In this sense, any nonlinear model could exhibit nonproportional changes in output given small changes in input under suitable parameter settings.

So-called Lyapunov exponents (see Appendix for technical details) are often used to characterize SDIC. The largest global Lyapunov exponent is usually taken to characterize the average growth rate of uncertainties in an infinitesimal set of initial conditions centered around $\boldsymbol{x}(0)$. These uncertainties represent all neighboring points surrounding $\boldsymbol{x}(0)$ from which trajectories could issue forth. The growth rate of these

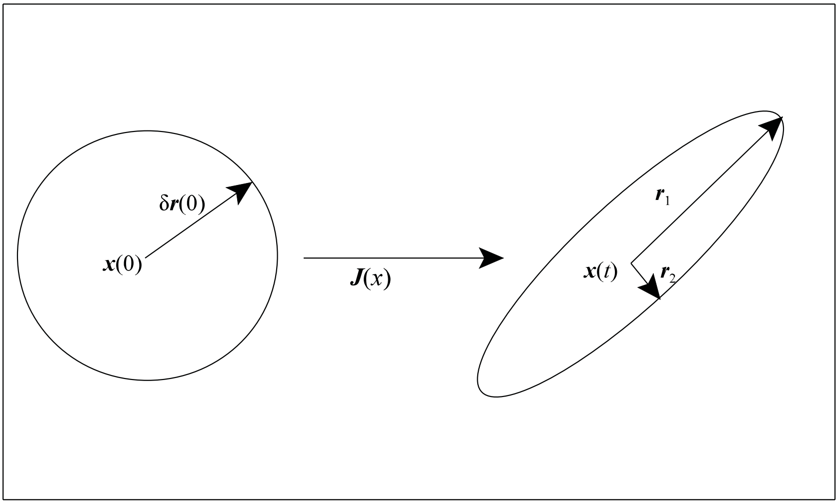

Figure 2. Evolution of an initially infinitesimal neighborhood of radius $\delta \boldsymbol{r}(0)$ of points in state space under the action of $\boldsymbol{J}(\boldsymbol{x})$. After some time t, the initial circle has evolved into an ellipse of semimajor axis $\boldsymbol{r}_1 = \delta \boldsymbol{r}(0)\exp(t\,\lambda_1)$ and semiminor axis $\boldsymbol{r}_2 = \delta \boldsymbol{r}(0)\exp(t\,\lambda_2)$.

uncertainties is standardly taken to be exponential. For simplicity consider two dimensions, so that there are only two global Lyapunov exponents. Given an initial infinitesimal circle of uncertainty of radius $\delta \boldsymbol{r}$ surrounding $\boldsymbol{x}(0)$ in state space, the image of the circle under the action of the propagator $\boldsymbol{J}(\boldsymbol{x}(t))$ is an ellipse.[3] In the limit of large time (i.e., infinitely long times), the global Lyapunov exponents are taken to characterize the time rate of growth (or shrinking) of the principal axes of the evolving ellipsoid, the semimajor axis being given by $\delta \boldsymbol{r}\exp(\lambda_1 t)$ and semiminor axis by $\delta \boldsymbol{r}\exp(\lambda_2 t)$, where $|\lambda_1| > |\lambda_2|$ (Figure 2). Geometrically, the quantity $\exp(\lambda t)$ represents the *average factor* by which the distance between trajectories issuing from neighboring initial points have spread after some time t (or have converged if λ is negative).

This latter feature is what lies behind the oft-stated claim that in nonlinear CM systems and models exhibiting chaotic dynamics, trajectories initiating from nearby points diverge exponentially fast from

3 $\boldsymbol{J}(\boldsymbol{x}(t))$ is a matrix (see Appendix) and can be thought of as a projector: It takes an uncertainty about some initial conditions at $t = 0$ and projects the future evolution of that uncertainty at later time t. See Figure 2.

one another. The on-average exponential growth rate of infinitesimal uncertainties is often taken to imply the exponential growth of small but finite uncertainties (e.g., Hobbs 1991, 155; Ott 2002, 140; Ruelle[4] 1994, 25 and 2004, 49; Smith 1998, 15).[5] Hence, the claim of exponential growth often gets translated into something like the following version of SDIC:

(SDIC) $\exists\lambda$ such that for almost all points $\boldsymbol{x}(0)$, $\forall\delta > 0\ \exists t > 0$ such that for almost all points $\boldsymbol{y}(0)$ in a small neighborhood (δ) around $\boldsymbol{x}(0)$ [$|\boldsymbol{x}(0) - \boldsymbol{y}(0)| < \delta$ and $|\boldsymbol{J}(\boldsymbol{x}(t)) - \boldsymbol{J}(\boldsymbol{y}(t))| \approx |\boldsymbol{J}(\boldsymbol{x}(0)) - \boldsymbol{J}(\boldsymbol{y}(0))|\, e^{\lambda t}$],

where the 'almost all' caveat is understood as applying for all points in the relevant state space except a set of measure zero. Here, λ is interpreted as the largest global Lyapunov exponent and is taken to represent the average rate of divergence of neighboring trajectories issuing forth from some small neighborhood centered around $\boldsymbol{x}(0)$. Exponential growth is implied if $\lambda > 0$ (convergence if $\lambda < 0$). In general, such growth cannot go on forever. If the system is bounded in space and in momentum, there will be a limit as to how far nearby trajectories can diverge from one another.

But there are problems with this way of defining SDIC for characterizing the dynamics of nonlinear systems their models. First, the definition of global Lyapunov exponents involves the infinite time limit, so, strictly speaking, λ only characterizes growth in uncertainties as t increases without bounds, not for any finite t. So the combination $\exists\lambda$ and $\exists t > 0$ in SDIC is inconsistent. At best, SDIC can only hold for the large time limit and this implies that chaos as a phenomenon can only arise in this limit. Furthermore, neither our models nor physical systems run for infinite time, but an infinitely long time is required in order to verify the presumed exponential divergence of trajectories issuing from infinitesimally close points in state space. It is often assumed that we can invoke the standard physicist's assumption that an infinite-time limit can be used to effectively represent some large but finite elapsed time. However, one reason to doubt this assumption in this context is that the calculation of finite-time Lyapunov exponents do not usually lead to on-average exponential growth as characterized by global Lyapunov

4 At least in popularizations.

5 Strictly speaking for Smith (1998), this implication applies to his discussion of what he calls (EXP), since he reserves SDIC for a much weaker definition (1998, 15).

exponents (e.g., Smith, Ziehmann and Fraedrich 1999). In general, for finite times $\boldsymbol{J}(\boldsymbol{x}(t))$ varies from point to point in state space (i.e., it is a function of the position $\boldsymbol{x}$ in state space and only approaches a constant in the infinite time limit), implying that the local finite-time Lyapunov exponents vary from point to point. Therefore, trajectories diverge and converge from each other at various rates as they evolve in time — the uncertainty does not vary uniformly in the chaotic region of state space (Smith, Ziehmann and Fraedrich 1999; Smith 2000). This is in contrast to global Lyapunov exponents which are on-average global measures of trajectory divergence and which imply that uncertainty grows uniformly (for $\lambda > 0$). Such uniform growth rarely occurs outside simple mathematical models (e.g., the baker's transformation).

For example, the Lorenz, Moore-Spiegel, Rössler, Henon and Ikeda attractors all possess regions where either (1) $\boldsymbol{J}$ has eigenvalues with negative real parts or (2) $\boldsymbol{J} + \boldsymbol{J}^{\mathrm{T}}$ has negative eigenvalues ($\boldsymbol{J}^{\mathrm{T}}$ is the transpose of the matrix $\boldsymbol{J}$). In the first case, eigenvalues with only negative real parts are not sufficient to rule out positive growth rates in uncertainties because in general $\boldsymbol{J}$ is non-normal (in addition to the eigenvalues, one also needs information about the projections of the uncertainty onto the eigenvectors and its decomposition into the eigenvectors). In numerical simulations, such regions are observed to be dominated by decreasing uncertainties in time. In the second case, no growth rates within such regions are positive. In either case, uncertainties shrink for the amount of time trajectories remain within such regions (e.g., Smith, Ziehmann and Fraedrich 1999, 2870-9; Ziehmann, Smith and Kurths 2000, 273-83). Hence, on-average exponential growth in uncertainties is not guaranteed for chaotic dynamics. Linear stability analysis can indicate when nonlinearities can be expected to dominate the dynamics, and local finite-time Lyapunov exponents can indicate regions on an attractor where *all* uncertainties will decrease so long as trajectories remain in those regions.

To summarize, the folklore that trajectories issuing forth from neighboring points in N always will diverge on-average exponentially in a chaotic region of state space is false.

The second problem with the standard account is that, there simply is no implication that finite uncertainties will exhibit an on-average growth rate characterized by any Lyapunov exponents, local or global. For example, the linearized dynamics used to derive global Lyapunov exponents presupposes infinitesimal uncertainties (Appendix (A1)-(A7)). But when uncertainties are finite, such dynamics do not apply and no valid conclusions can be drawn about the dynamics of finite uncertainties from the dynamics of infinitesimal uncertainties. Certainly infinitesimal uncertainties never become finite in finite time — barring super exponential growth. Even if infinitesimal uncertainties became finite after a finite time, that would presuppose the dynamics is *unconfined*, whereas chaotic dynamics is usually characterized as being confined to some attractor (strange attractor in the case of dissipative systems, the energy surface in the case of Hamiltonian systems[6]). The interesting features of nonlinear dynamics usually take place in subregions of state space, so presupposing an unconfined dynamics would be inconsistent with the features we are trying to capture.

Can the on average exponential growth rate characterizing SDIC ever be attributed legitimately to diverging trajectories if their separation is no longer infinitesimal? Examining simple models might seem to indicate yes. However, answering this question requires some care for more complex systems like the Lorenz equations. It may turn out that the rate of divergence in the finite separation between two nearby trajectories in a chaotic region changes character numerous times over the course of their winding around in state space, sometimes faster, sometimes slower than that calculated from global Lyapunov exponents, sometimes contracting, sometimes diverging (Smith, Ziehmann and Fraedrich 1999; Ziehmann, Smith and Kurths 2000). But in the long run, some of these trajectories can *effectively* diverge *as if* there were on-average exponential growth in uncertainties as characterized by global Lyapunov exponents.[7] The details of the kinds of divergence (convergence) neighboring trajectories undergo turn on the detailed structure of the dynamics (i.e., it is determined point-by-point by local growth and convergence of finite uncertainties under the action of $\boldsymbol{J}(\boldsymbol{x}(t))$ and not by any Lyapunov exponents).

But as a practical matter, all finite uncertainties saturate at the diameter of the attractor. This is to say, that the uncertainty reaches some maximum amount of spreading after a finite time and is not well-quantified by global measures derived from Lyapunov exponents (e.g., Lorenz 1965). So the folklore — that on-average exponential divergence of trajectories characterizes chaotic dynamics — is misleading for most nonlinear models and systems. Therefore, drawing an inference from the presence of positive global Lyapunov exponents to the existence of on-average exponentially diverging trajectories is invalid. This has

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6 A Hamiltonian system is one where the total kinetic energy plus potential energy, is conserved. In contrast, dissipative systems lose energy through some dissipative mechanism such as friction or viscosity.

7 However, it is conjectured that the set of initial points in the state space exhibiting this behavior is a set of measure zero, meaning, in this context, that although there is an infinite number of points exhibiting this behavior, this set represents zero percent of the number of points composing the attractor.

implications for the SD argument discussed below (§IV.3.2) as well as for characterizing chaos.

2. Chaos

Although there is no agreement on what constitutes necessary and sufficient conditions for defining chaos, there does exist a wide measure of agreement as to the necessary features characterizing a dynamics as chaotic (e.g., Batterman 1993; Kellert 1993; Hilborn 2001; Smith 1998):

(a) Trajectories are confined due to some kind of stretching and folding mechanism.[8]

(b) Some trajectory orbits are *aperiodic*, meaning that they do not repeat themselves on any time scales.

(c) Trajectories exhibit SDIC.

Of these three features, SDIC is often taken to be crucial and is often suspected as being related to the other two. That is to say, SDIC is a property of a particular kind of dynamics that can only exist in nonlinear systems and models. As discussed above, however, if SDIC is characterized by on-average exponential growth, then (a) - (c) at best only pick out those cases where there is on-average exponential growth of infinitesimal uncertainties (e.g., simple cases like the baker's transformation, where $\boldsymbol{J}$ is constant even for finite times), which are both rare and of little practical interest.

In view of §II.1, then, it seems more appropriate to replace SDIC as defined above with a property that exhibits the characteristic feature of nonlinear response, namely that a small change in a parameter or variable leads to a large (nonproportional) change in output. Call this alternative condition SDIC*. This condition should be formulated such that it does not presuppose the infinite time limit, nor presupposes only infinitesimal quantities. Furthermore, it should rule out cases of small (linear) growth in uncertainties so as not be vacuously fulfilled by linear growth for very large times. It would not, however, indicate that growth in uncertainties would always be exponential. Unfortunately, we currently do not know how to formulate such a condition, though if one was available, we could then say that a dynamics is chaotic just in case it exhibits (a), (b) and SDIC*.

8 Recall that this confinement need not be due to physical walls of some container. If, in the case of Hamiltonian chaos, the dynamics is confined to an energy surface, this surface could be spatially unbounded.

III Unique Evolution

The models of CM are often taken to be paradigm cases of the type of determinism espoused by the Laplacean vision[9] which has the following three components (Stone 1989; Kellert 1993, chapter 2; Bishop 2003 and 2005):

(DD) Differential Dynamics: An algorithm relates a state of a system at any given time to a state at any other time and the algorithm is not probabilistic.

(UE) Unique Evolution: A given state is always followed (or preceded) by the same history of state transitions.

(VD) Value Determinateness: Any state can be specified with arbitrarily small (nonzero) error.

Differential dynamics is motivated by actual physical theories expressed in terms of nonprobabilistic mathematical equations, ICs and BCs, expressing the Laplacean belief that there are no indeterministic elements in CM like those present in some versions of quantum mechanics. Unique evolution is closely associated with DD and expresses the Laplacean belief that the systems and models of CM will repeat their behaviors exactly if the same ICs and BCs are specified. For example the equations of motion for a frictionless pendulum will produce the same solution for the motion as long as the same initial velocity and initial position are chosen. Value determinateness is motivated by the Laplacean belief that there is nothing *in principle* in CM preventing mathematical descriptions of arbitrary accuracy. For example the models of CM all presuppose precise values for the constants and variables used in the equations of motion. It is only with the advent of quantum mechanics (QM) that questions were raised about the applicability of value determinateness to all of physics.[10]

As far as the SD argument is concerned, UE is the crucial element of determinism. Here is one way to explicate UE: Let *S* stand for the col-

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9 Suitable changes must be made to accommodate continuum mechanics, but I will ignore these differences here.

10 Historically a fourth property known as absolute predictability completed the picture of determinism as conceived by Laplace, but the relationship of predictability to determinism is more subtle than typically realized and the type of predictability implied by DD, UE and VD is also much weaker than often conceived (Bishop 2003).

lection of all systems sharing the same set *L* of physical laws and suppose that *P* is the set of relevant physical properties for specifying the time evolution of a system described by *L*:

> A system $s \in S$ exhibits *unique evolution* if and only if every system $s' \in S$ isomorphic to *s* with respect to *P* undergoes the same evolution as *s* (compare Bishop and Kronz 1999, 130-1).

Roughly the idea is that every time we return the system to the same initial state, it will undergo the same history of transitions from state to state. Imagine a typical physical system as analogous with a film. Unique evolution means that if we were to start the film over and over at the same frame (returning the system to the same initial state), then the film would repeat every frame in sequence over and over again, and identical copies of the film would produce the same sequence of frames. So if we always start *Jurassic Park* at the beginning frame, it plays the same. No new frames are added to the movie and the sequence of frames never varies. In other words the evolution of the system will be unique with respect to a particular specification of the ICs and BCs and, furthermore, we can choose *any* state in the history of state transitions as the initial starting point and there will still be a unique history of state transitions. The 'collection' of systems is just all the isomorphisms allowed by the properties of the equations of motion.

IV The SD Argument

Now we are in a position to assess whether or not QM threatens determinism in classical nonlinear system. Hobbs (1991, 157) and Kellert (1993, 69-75) both give versions of SD arguments to the effect that quantum events influence CM systems exhibiting SDIC. Crucial to these arguments is the idea that UE fails for such systems when QM is taken into account. The SD argument may be succinctly formulated as follows (compare with Bishop and Kronz 1999, 134)[11]:

11 The arguments in this section are framed in terms of SDIC since the literature discussing the SD argument has been framed this way to date. At the end of §IV I will discuss what comes of the SD argument when this assumption is modified as it must be in light of §II.

(A) For nonlinear systems exhibiting SDIC, trajectories starting out in a highly localized region of state space will diverge on-average exponentially fast from one another.

(B) Quantum mechanics limits the precision with which physical systems can be specified to a neighborhood in state space of no less than $(2\pi/h)^N$, where h is Plank's constant (with units of action) and N is the dimension of the system in question.[12]

(C) Given enough time and the QM bound on the neighborhood ε for the ICs, two trajectories of the same chaotic nonlinear system will have future states only localizable to a much larger region δ in state space (from (A) and (B)).

(D) Therefore, QM will influence the outcomes of chaotic systems leading to a violation of UE.

Briefly, the reasoning runs as follows. Two isomorphic nonlinear systems of CM exhibiting SDIC, whose initial states are localized within ε, will have future states that can be localized only within $\delta >> \varepsilon$. Since QM sets a lower bound on the size of the patch of ICs, UE must fail for nonlinear chaotic systems.

1. Hobbs' Defense of the SD Argument[13]

Hobbs (1991, 157) supports the SD argument by seeking to demonstrate that nonlinear systems exhibiting SDIC cannot avoid the indeterminism of QM. To evaluate his case, recall the distinction between Hamiltonian and dissipative systems. In the latter case, the state space volume in which trajectories evolve contracts. In addition, the dynamics of nonlinear dissipative systems can develop peculiar geometric structures called *strange attractors* due to the stretching and folding of state space trajectories. It is important to clarify that the trajectories do not stretch and fold because there is a strange attractor; rather, strange attractors form *as a result* of the stretching and folding of trajectories.

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12 This is the precision limit for the state of minimum uncertainty for measuring momentum and position pairs in an N-dimensional quantum system (note, the exponent is $2N$ in the case of measuring uncorrelated electrons).

13 Kellert does not offer much in the way of a defense of the SD argument, but puts the argument to use in an attempt to demonstrate that DD is the core feature of determinism. For critical discussion of this thesis see (Bishop 2002b, 26-8 and Bishop 2005a).

Strange attractors normally only occupy a subregion of state space, but once a trajectory wanders close enough to the attractor, it normally approaches ever closer to the surface of the attractor for the rest of its future. Strange attractors are fractals (characterized by *noninteger dimensionality*), but the key feature for Hobbs is that they possess *self-similar structure*. That is to say, if we were to magnify any small portion of a strange attractor, we would find that the magnified portion looked identical to the original region. If we were to magnify the magnified region again, we would see the identical structure repeated again. Continuous repetition of this process would yield the same results. The self-similar structure is repeated on all scales in state space. In other words, strange attractors have infinite layers of repetitive structure (at least for our mathematical models). This type of repetitive structure arises as trajectories remain within a subregion of state space by folding and intertwining with one another without ever intersecting or repeating themselves exactly (what I referred to as confinement earlier).

Hobbs argues one of two possibilities will follow from the fact strange attractors possess self-similar structure on all scales: either (i) the self-similar structure approaches closer and closer to the quantum level, thus increasing the prospects for quantum effects to penetrate the macroscopic system, or (ii) there is some level below which self-similar structure does not apply, so that the quantum level is reached (1991, 157). Either way, UE would fail as indicated by the SD argument. For Hobbs, the presence of pervasive self-similar structure, then, 'suggests' that quantum indeterminism can 'scale up' to the macroscopic level in a relatively short time.

The conclusion Hobbs wants to draw from (ii) is that quantum effects cause UE to breakdown. Such a conclusion is problematic, however. Nonlinear dissipative systems of equations exhibiting strange attractors have repeating self-similar structure on arbitrarily small scales. However, things are trickier here than Hobbs allows. First, physical systems tend to only exhibit quasi-strange attractors with *prefractal* geometries, i.e., with only two or three repetitions of self-similar structure (Avnir et al. 1998). On such a possibility, quantum mechanics is irrelevant as Hobbs understands things because what lies below the last repetition is, again, a classical state space and classical trajectories.

On the other hand, it is possible that there are strange attractors with countably infinite repetitions of self-similar structure, but why should this lead to a problem with QM? Here, we see that Hobbs' reasoning in (ii) depends upon (i), but the conclusion in (i) rests on a fundamental confusion. The self-similar structure of strange attractors is a feature of *state space* and not a feature of *physical space*. Just because a trajectory of a system in state space is spiraling ever closer to a strange attractor does not imply that the trajectory in physical space is somehow ever more closely approaching the quantum level as the state space trajectory approaches the attractor. Think of a chaotic pendulum swinging in space; as the pendulum traces out its chaotic trajectory, there is no sense in which its trajectory approaches the quantum realm. So the existence of self-similar structure in some region of state space does not imply that physical trajectories of the system somehow become more susceptible to quantum effects. After all, they are simply spiraling around forming more and more intricate structures through stretching and folding processes, not approaching nearer and nearer to the quantum realm.[14]

2. Criticisms of the SD Argument

Fortunately, the formulation of the SD argument given above does not require the kind of defense Hobbs offers. Rather, I take it that the key idea is that the smallest of differences may be amplified over time by chaotic dynamics in nonlinear systems. And a quantum fluctuation certainly counts as a small difference.

However, there is an obvious objection to this intuitive idea behind the SD argument. To have any force against UE, the argument must presuppose that the isomorphic systems are in identical initial states, but differ in their history of state transitions. Under some interpretations of QM, however, the concept of identical states or identical sets of ICs is problematic because there are no precise values of the ICs of the two chaotic systems upon which to agree. The fundamental problem is that the success of the SD argument depends crucially on what view of QM and the measurement problem is adopted.[15]

This can be seen clearly in the following idealized example. Consider a damped, driven pendulum where the driving frequency, amplitude and coefficient of friction are such that the pendulum exhibits SDIC. Now suppose we fire a single photon at the pendulum and that this photon strikes the pendulum arm, imparting some amount of momentum to the arm. Weak as the photon's contribution may be, this tiny addition to the momentum of the chaotic pendulum's arm is different from what the momentum of the arm would have been in the absence

14 There are technical complications that arise in the mathematical analysis of how QM and its state space affect the state space representation of a nonlinear CM model when the two are co-joined (for discussion, see Bishop and Kronz 1999, 135-6).

15 Neither Hobbs nor Kellert explicitly states a preferred interpretation on these issues, although Hobbs seems to have some form of Copenhagen interpretation in mind.

of the photon.[16] And, according to the SD argument, this difference in ICs is enough to yield a difference in the macroscopic behavior of the chaotic pendulum's motion at some point in the future relative to the undisturbed pendulum.

In QM, the state vector contains all the information about a given quantum system. State vectors are specified as vectors in Hilbert space, where the tip of the vector is a point traveling on the surface of a unit hyper-sphere. Given the same initial preparation of the quantum state, such vectors will undergo the same Schrödinger evolution (the tip of the arrow will trace out the same path) until interacting with some system. For the purposes of assessing the SD argument, I will treat the pendulum as if it can be fully analyzed in terms of CM (although we will see below that friction must be treated with a mixed model combining both classical and quantum features). It is reasonable to assume that the moment at which the chaotic pendulum registers the quantum fluctuation, a measurement has occurred because an irreversible act of amplification of the photon's transfer of momentum to the pendulum has taken place. According to the SD argument, because of SDIC the photon's contribution of momentum will be amplified such that it affects the motion of the pendulum.

For versions of QM in which measurement processes do not indeterministically collapse the state vector (i.e., in which there is only Schrödinger evolution), there will be no indeterminism at the quantum level for sensitive dependence to amplify.[17] Whereas on other versions of QM, measurement processes do lead to indeterministic collapse of the state vector to one sharp value from the superposition of possible values represented by the state vector. The indeterministic collapse results in potential violations of UE.

The concept of identical sets of ICs under indeterministic collapse interpretations for the pendulum plus photon system is, however, ambiguous. Specifying the pendulum plus photon system before the interaction takes place as the initial state (Figure 3), leads to the same ICs every time the experiment is run (provided the initial pendulum and photon states are prepared identically). This is because the photons start out in the same pure state representing a superposition of

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16 Although a conceptually simple example, the electromagnetic interaction between the photon and pendulum is actually quite complex. One could also think of sending silver atoms through a Stern-Gerlach apparatus, where an atom of either spin up or spin down strikes the pendulum's arm, a less complex interaction. The following analysis, however, remains unchanged.

17 Similarly, there would be no indeterminism to amplify on deterministic theories of QM such as that developed by David Bohm.

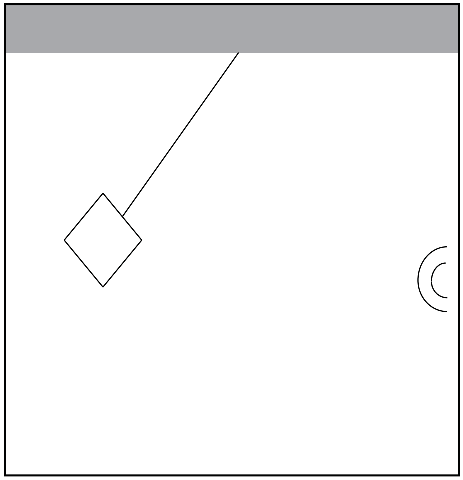

Figure 3.
Pendulum+photon system.

possible energy and frequency eigenstates and it is this state vector *as a whole* plus the state of the pendulum that are taken as the ICs. Under this specification the system is indeterministic due to the later indeterministic collapse of the photon state vector as a result of the measurement-like interaction between the photon and the pendulum. Unique evolution fails for this case because, although the system always starts in the same initial state, the photon state vector collapses indeterministically at a different location on the pendulum transferring a different amount of momentum each time the experiment is run producing different trajectories for the pendulum. Hence we have the same initial states, but different future states which explicitly violates UE.

On the other hand specifying the moment of interaction as the initial state (Figure 4) results in different ICs every time the experiment is run. This amounts to treating the photon as an indeterministic perturbation to the motion of the pendulum. The photons may be prepared in the same initial state each time the experiment is run, but an indeterministic change in the pendulum's position and momentum state results because of the indeterministic collapse of the photon's state vector upon interacting with the pendulum arm. There would then be a different initial condition (a different indeterministic perturbation) for the pendulum's motion.[18] Hence we have different initial states and different

18 There is a continuum of possible position and momentum states for the photon that could be actualized in interaction with the pendulum. Therefore, the probability of achieving the same ICs on multiple runs under this second specification is of measure zero.

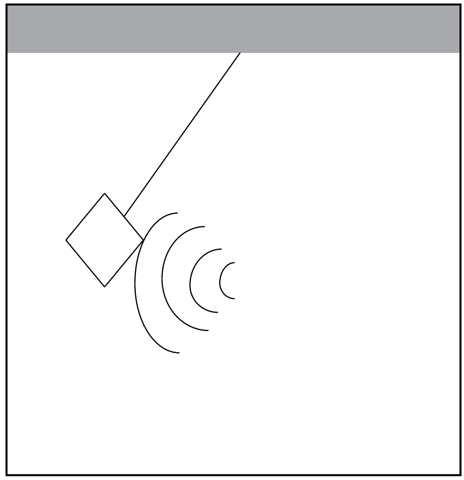

Figure 4.
'Moment of interation.'

future states which is consistent with UE. Therefore the purported failure of UE also depends on how one 'cuts up' the system in cases where the quantum state vectors collapse indeterministically.

This line of argument is related to a fundamental problem of QM noted by John Bell:

> The problem is this: quantum mechanics is fundamentally about 'observations'. It necessarily divides the world into two parts, a part which is observed, and a part which does the observing. The results depend in detail on just how this division is made, but no definite prescription for it is given. All we have is a recipe which because of practical human limitations is sufficiently unambiguous for practical purposes. (1987, 124)

Essentially QM allows several different ways of 'cutting up the system' between observer and observed. For the case of our pendulum and photon, the first specification of the initial state corresponds to choosing to observe the photon+pendulum system as a whole. The second specification corresponds to the pendulum as measuring apparatus that in turn responds to the perturbation imparted by the detected photon. So the question of choice of ICs is related to the problem of how to divide the system into observer and observed.

Table 1 broadly summarizes the upshot of the simple photon plus pendulum experiment for the various versions of QM and the specification of initial states (cases of deterministic state vector collapse will have the same results as Schrödinger evolution alone).

Granted the photon plus pendulum experiment is very simplified and adding complications will change these results. Consider a slightly more complicated case, where in addition to the initially prepared

Table 1. The effects of various versions of QM, measurement theories and specifications of ICs on unique evolution.

Quantum Process	Unique Evolution for Photon + Pendulum System as IC	Unique Evolution for Moment of Interaction as IC
Schrödinger Evolution	Preserved	Preserved
Schrödinger + Indeterministic Collapse	Violated	Preserved

photon, we add a photon prepared well after the first. Once the initial photon has been registered by the pendulum, any further interactions between the pendulum and the second photon lead to violations of UE, unless all state vector collapses are deterministic. One might attempt to account for the effect of the second photon (or any further ones) by means of a BC, but the net effect of this move is to render CM irreducibly indeterministic by loading indeterminism into the BC (and violating condition DD above).

Kellert shows some sensitivity to the general problem of identifying identical ICs in QM, but misses the point by appealing to the possibility of quantum chaos as the source of the confusion on identical initial quantum states (1993, 72-3). The status of quantum chaos is independent of the present concerns about how quantum systems interact with nonlinear CM systems, being largely the study of how chaos emerges in quantum systems (if it does at all) and the study of quantized versions of classically chaotic systems. A potentially more relevant worry is raised by Wojciech Zurek (1998), who argues that QM actually tames the radical growth in uncertainty chaotic CM systems like the orbit of Hyperion would exhibit in the absence of chaos. Unfortunately, Zurek's analysis is seriously flawed. He invokes the problematic global Lyapunov exponents to generate the supposed radical growth in uncertainty. Moreover, he relies on the 'correspondence limit' where Planck's constant is imagined to go to zero. Physically, this is impossible since, as a universal constant, it cannot physically change its value. The kind of limit Zurek must have in mind is some ratio involving the classical action and Planck's constant, which becomes singular in the classical limit. There actually is no classical limit of QM that is nonsingular, meaning that the relationship between QM and CM is much more complex than Zurek's analysis admits (Primas 1998; Bishop 2005b). Hence, the status of quantum chaos is irrelevant for SD arguments.[19]

3. Where the Rub Is

The upshot of §IV.2 is that *in principle* quantum effects can be amplified via sensitive dependence and, in turn, influence the dynamics of nonlinear chaotic systems. The upshot for determinism in nonlinear CM systems is less clear, however. In cases where quantum effects are amplified to the macroscopic level via SDIC for situations like the damped, driven pendulum, the question of whether such effects lead to violations of UE depends upon the version of QM, the solution to the measurement problem and, in concrete cases, what counts as the initial conditions for the quantum plus nonlinear CM system under investigation. But how plausible is it that such a thing could happen? Are there limits on whether or not quantum effects could be amplified by something like a chaotic pendulum? Yes. There are at least two constraints that may seriously limit the amplification of such effects.

3.1. Limitations Due to Damping

In dissipative systems, like the damped driven pendulum, mechanisms such as friction and diffusion certainly will place lower limits on the magnitude quantum effects must have to influence nonlinear CM systems. The exciting feature of chaotic dynamics is that very simple nonlinear systems yield extremely complex behavior. What has been neglected in the SD argument so far, however, is the *detailed character* of dissipative mechanisms necessary to produce chaotic dynamics in dissipative systems. An examination of features such as sliding friction shows that once again the SD argument does not go through as straightforwardly as Hobbs and Kellert assume.

Damping due to friction at the pivot point is a necessary condition for chaotic behavior of pendulums, but the simplicity of our models for such systems overlooks the fact that sliding friction is a complex wide-scale phenomenon. Typically, the coefficient of friction in the pen-

19 Barone *et al.* (1993) have performed computer simulations of classical electrons orbiting in the field of a charged sphere with a uniform magnetic field superimposed. They argue that for parameter settings leading to chaotic dynamics, macroscopic indeterminism results from the fact that QM limits the precision available in state space for specifying ICs. Their approach, however, is subject to the same problems as those I have identified for the sensitivity arguments put forward by Hobbs and Kellert.

dulum's equation of motion is treated as a phenomenological constant, presumably representing some bulk or averaged effects. Such a friction term looks to preclude any possibility of quantum effects entering into the dynamics of the pendulum — they should be damped out if they are even physically relevant. The phenomenological constant suggests they are not physically relevant, so why does this not put an end to the SD argument at least in the case of the photon+chaotic pendulum system? The following discussion indicates that frictional processes are much more complicated than this simple phenomenological treatment. And it is the details of frictional processes that possibly opens the door for quantum effects to be amplified by the dynamics of a chaotic pendulum because our best accounts of friction involve microscopic features not taken into account in our mathematical models of pendula.

At microscopic scales the surface of even highly polished materials is far from smooth. Rather, surfaces resemble a jumble of jagged 'peaks and valleys,' with typical peak-to-valley distances being of the order of several thousand atomic diameters. Therefore, the actual microscopic area of contact between two surfaces can easily be 10,000 times smaller than the apparent macroscopic area of contact. Sliding friction, then, is due to many possible processes giving rise to the resistance and heating with which we are familiar. One of the 'peaks' of one surface can plough a 'valley' or 'peak' of another surface leading to shearing or deformation. Macroscopic flaws of the two surfaces can interlock. Also the 'peaks' of one surface can form molecular bonds with the 'peaks' of another surface due to intermolecular forces. When one metal surface is pulled across the surface of another, myriads of these bonds are constantly being ripped apart while new ones are being formed.

Presumably what is needed for the chaotic pendulum is for quantum effects, like the interaction with a photon, to cause some bonds to form (or break) that would not normally form (or break) on a time scale due purely to the macroscopic mechanical process of two surfaces sliding across one another. It has been shown that for three degrees of freedom, models of frictional vibration due to stick-slip motion can exhibit chaotic behavior (Bengisu and Akay 1992, 562-5). So, suppose the smallest possible difference is the delay in the formation or breaking of one bond. Then, according to the SD argument, this is sufficient to eventually influence the trajectory of the chaotic pendulum through SDIC.

Of course, the key worry is that quantum effects are too minute to play any appreciable role in friction processes. It turns out, however, there are several quantum effects associated with valence electrons on metal surfaces that contribute significantly to macroscopic fictional processes. These electrons move about freely like a gas producing the forces responsible for molecular bonds between two metal surfaces.[20] Typical molecular bonds due to interatomic potentials have strengths on the order of 10^{-13} g·cm/sec^2, so this would place a lower bound on the magnitude of quantum effects needed to break such a bond for so-called dry surfaces, i.e., where there are no contaminants such as oxides or lubricants on the metal surfaces. The extent to which free electrons penetrate into the vacuum above the surface of a metal is one Angstrom (10^{-12}cm). Therefore, a film of oxide or lubricant need only be about three or four angstroms thick to screen off Coulomb quantum effects due to Pauli's exclusion principle (roughly, a principle that prevents two electrons from occupying the same state). Contaminated surfaces, which are the usual case in practical situations, *would not* prevent any quantum effects in the friction process due to tunneling or dipole interactions.

In many practical situations, not only will there be a thin film or oxide layered on the surface of metals, but after some amount of time, metal surfaces rubbing against one another will produce a situation where larger-scale effects like ploughing will become negligible with respect to the interfacial effects such as those due to phonons and conduction electrons (Mak, Daly and Krim 1994, 190). Using semi-classical calculations, Bo Persson (1991) has argued that effects due to electrons make the dominant contributions to the forces governing friction at the molecular level under these circumstances.[21] Experiments (Mak, Daly and Krim, 1994) and derivations from first principles (Sokoloff 1995) support Persson's models.

So quantum effects play a non-negligible role in the processes leading to the formation and maintenance of molecular bonds in friction processes.[22] What is the upshot for our photon+chaotic pendulum system? *If a perturbation due to the photon can be amplified to a sufficient magnitude on a short enough time scale, in principle it could affect an intermolecular bond*. Presumably all that is needed for the SD argument is for the photon's additional momentum to force a single bond to break, say, a fraction of an instant sooner than what would normally take place

20 Some examples of such processes are electron tunneling (Persson and Demuth 1985), electron-hole pair production and fluctuations (Persson and Zaremba 1985; Persson 1991), and vibrational interactions through dipole fields (Persson and Ryberg 1981) among others.

21 See (Tobin 1993) and (Persson 1993a and 1993b) for objections to and a defense of these calculations.

22 Irwin Singer (1994, 2613-14) has succinctly summarized a number of molecular mechanisms at work in friction. Such mechanisms provide opportunities for quantum effects to play a non-negligible role in the dynamics of friction.

(due to mechanical or other processes) in the absence of the photon's impact on the chaotic pendulum's arm. Accordingly, the SD argument indicates the effects of this seemingly insignificant event would grow in time to influence the classical trajectory traced out by the pendulum. Admittedly, the constraints would be very difficult for a photon to surpass even in our idealized photon+chaotic pendulum system. However, as the analysis in this section indicates, there are plenty of other quantum effects that could act such that they are amplified by the chaotic pendulum.

One might still argue that the quantum effects like the photon's perturbation will be 'swamped out' due to the fact that they are exceedingly tiny with respect to classical effects. It is not clear how such quantum effects would be swamped out given SDIC, however. One might think that macroscopic effects will always dominate a quantum perturbation because the former will always be larger in magnitude. Obviously photons and electrons are minute compared to a pendulum, but there is plenty of evidence that effects due to electrons at least contribute significantly to the physics of sliding surfaces (Singer 1994). Furthermore, in the context of chaotic nonlinear dynamics, magnitude should not be confused with significance due to the prospects of amplification of small effects. One might assume that quantum perturbations will cancel out to zero; however, the experimental evidence indicates otherwise (Mak, Daly and Krim 1994). If the cancellation is even minutely different from zero, chaotic dynamics can amplify this difference. The key question is whether this amplification can happen quickly enough.

3.2 Limitations Due to Nonlinear Dynamics

Another possible constraint on the SD argument comes from its reliance upon global Lyapunov exponents. As formulated in §IV.2 (following Hobbs and Kellert), the argument depends on SDIC, which presupposes both infinitesimal uncertainties in ICs as well as on-average exponential growth for such uncertainties throughout the duration of the dynamics. But as pointed out in §II.2, infinitesimal effects cannot grow to finite size in finite time, so if the photon contributes only an infinitesimal change to the pendulum's momentum, there will be no opportunity for its contribution to be amplified over the finite life-time of the pendulum's motion. The SD argument would then fail.

In order to save the SD argument then, SDIC* must be substituted in (A) as the appropriate characterization of the growth of uncertainties instead of SDIC. The jump the photon adds to the pendulum's momentum surely is finite though small (violating one of the assumptions underlying SDIC). Assume for the sake of argument that it is of sufficient magnitude to be amplified up and break a molecular bond perhaps on a time scale shorter than the mechanical action of the pendulum would break the bond. There are now several possibilities for the photon's contribution. Given the structure of the dynamics of sliding friction and the fact that the growth in uncertainties is characterized by the nonlinear local point-by-point dynamics (e.g., in the attractor region in question, either $\boldsymbol{J}$ has eigenvalues with negative real parts or $\boldsymbol{J} + \boldsymbol{J}^T$ has negative eigenvalues as discussed in §II.2), the uncertainty in momentum generated by the photon (1) may grow slower than is necessary to break a bond on the appropriate time scale, (2) may converge rather than grow, or (3) may grow rapidly enough to break the bond before mechanical action does so. The first two possibilities would preclude the photon from having any effect on the pendulum's evolution in which event the SD argument would fail.

V Concluding Remarks

Many authors have argued that chaos opens a door for QM to 'infect' CM (e.g., Hobbs 1991; Barone *et al*. 1993; Kellert 1993; Bishop and Kronz 1999). However, as a point of clarification, it is not chaos per se that opens this door. Rather, it is the nature of particular kinds of nonlinear dynamics — those which exhibit stretching and folding (confinement) of trajectories, where there are no trajectory crossings, and which exhibit aperiodic orbits — having the property of SDIC* that open the door for quantum effects. However, as the analysis given here indicates, the SD argument does not go through as smoothly as advocates have thought. There are difficult issues regarding the appropriate version of QM, the nature of quantum measurement theory, and the selection of the initial state characterizing the system that must be resolved before one can say clearly whether or not UE is violated. Furthermore, the possible constraints of nonlinear CM systems on the amplification of quantum effects must be considered on a case by case basis such as damping and the local finite-time dynamics for each system. This is to say, that there is no abstract, a priori reasoning establishing the truth of an SD argument; the argument can only be demonstrated on a case-by-case basis.[23]

In addition, the question as to whether such quantum interactions with nonlinear CM systems exhibiting SDIC* contribute *indeterministically* to the outcomes of such CM systems depends on the currently

23 Perhaps detailed examination of enough cases would allow us to make some generalizations about how wide-spread the possibilities for the amplification of quantum effects are. This would be a worthwhile project.

undecidable question of indeterminism in QM and the measurement problem. There is a serious open question as to whether the indeterminism in QM is simply the result of ignorance or if it is an ontological feature of the quantum world. Suppose that QM is ultimately deterministic, but that there is some additional factor, a hidden variable as it is often called, such that if this variable were available to us, our description of quantum systems would be fully deterministic. Another possibility is that there is an interaction with the broader environment that accounts for how the probabilities in quantum mechanics arise (physicists call this approach 'decoherence'). Under either of these possibilities, we would interpret the indeterminism observed in QM as an expression of our ignorance, and, hence, indeterminism would not be a fundamental feature of the quantum domain. It would be merely *epistemic* in nature due to our lack of knowledge or access to quantum systems. So if the indeterminism in QM is not ontologically genuine, then whatever contribution quantum effects make to CM systems exhibiting SDIC* would not violate UE.

In contrast, suppose it is the case that such QM is genuinely indeterministic; that is, all the relevant factors of quantum systems do not fully determine their behavior at any given moment. Then the considerations in this essay would indicate that not all physical systems traditionally thought to be in the domain of CM can be described using strictly deterministic models, leading to the need to approach the modeling of such nonlinear systems differently (e.g., Bishop and Kronz 1999, 138-9). Furthermore, as the discussion of quantum processes in friction indicate (§IV.3.1), it is plausible that quantum effects could be interacting with and influencing nonlinear CM systems more often than we are used to considering. A necessary, though by no means sufficient, condition for this interaction is that nonlinear systems exhibit SDIC*. This has interesting implications for the foundations of nonlinear dynamics and the interface between QM and CM, as well as for intriguing questions regarding how quantum effects might be related to free will and consciousness if neural dynamics also exhibits SDIC* (Kane 1996; Bishop 2002a). This essay represents an opening foray into such implications.

Appendix

One way to get a handle on Lyapunov exponents is to see how they arise out of linear stability analysis of the trajectories of evolution equations. Consider the first-order, ordinary differential equation system $d\boldsymbol{x}/dt = \boldsymbol{Fx}$ and suppose that $\boldsymbol{x}^*$ is a steady point, i.e. a point at which $\boldsymbol{F}(\boldsymbol{x}^*) = 0$. We can study the behavior of trajectories near $\boldsymbol{x}^*$ by considering $\boldsymbol{x}(t) = \boldsymbol{x}^* + \varepsilon(t)$, where $\varepsilon(t)$ is an infinitesimal perturbation to every component of $\boldsymbol{x}$. Substituting back into $\boldsymbol{F}$ and expanding to first order in $\varepsilon(t)$ yields

$$\boldsymbol{F}(\boldsymbol{x}^* + \varepsilon) = \boldsymbol{F}(\boldsymbol{x}^*) + \boldsymbol{J}(\boldsymbol{x}^*)\varepsilon + O(\varepsilon^2) \ , \qquad \textbf{(A1)}$$

where the matrix $\boldsymbol{J}(\boldsymbol{x})$ is the $n \times n$ Jacobian matrix of partial derivatives of $\boldsymbol{F}$ evaluated at the point $\boldsymbol{x}$, and the explicit dependence on t has been suppressed. We then obtain an equation for the time dependence of the perturbation of $\boldsymbol{x}$, namely

$$\frac{d\varepsilon}{dt} = \boldsymbol{J}(\boldsymbol{x}^*)\, \varepsilon + O(\varepsilon^2) \ . \qquad \textbf{(A2)}$$

A linear stability analysis results if we neglect terms of ε^2 in (A2). If ε is a real-valued vector and $\boldsymbol{J}$ a real-valued matrix (i.e., having no complex values), and we assume a solution of the form $\varepsilon = \lambda e^{st}$, (A2) reduces to the eigenvalue equation

$$\boldsymbol{J}\lambda = s\lambda \ . \qquad \textbf{(A3)}$$

Linear stability analysis is used to characterize Lyapunov exponents for nonlinear systems of equations. Consider the initial condition $\boldsymbol{x}(0)$ for our first-order system of differential equations and an infinitesimal displacement from $\boldsymbol{x}(0)$ in the direction of some tangent vector, $\boldsymbol{y}(0)$. Then the evolution of $\boldsymbol{y}$ according to (A2) is given by

$$\frac{d\boldsymbol{y}}{dt} = \boldsymbol{J}(x)\boldsymbol{y} \ . \qquad \textbf{(A4)}$$

valid for only an infinitesimal neighborhood about $\boldsymbol{x}(0)$. So the value of the vector $\boldsymbol{y}$ changes in time according to the values $\boldsymbol{J}$ takes on over time. Here $\boldsymbol{y}/|\boldsymbol{y}|$ gives the direction of the infinitesimal displacement from $\boldsymbol{x}$, where the bars indicate absolute magnitude. Additionally, $|\boldsymbol{y}|/|\boldsymbol{y}(0)|$ gives the factor by which the infinitesimal displacement grows ($|\boldsymbol{y}| > |\boldsymbol{y}(0)|$) or shrinks ($|\boldsymbol{y}| < |\boldsymbol{y}(0)|$). The Lyapunov exponent is now defined with respect to initial condition $\boldsymbol{x}(0)$ and initial orientation of the infinitesimal displacement $\boldsymbol{y}(0)/|\boldsymbol{y}(0)|$ as

$$\lambda(\boldsymbol{x}(0), \frac{\boldsymbol{y}}{|\boldsymbol{y}(0)|}) = \lim_{t\to\infty} \frac{1}{t} \ln\left(\frac{\left|\frac{d\boldsymbol{y}}{dt}\right|}{|\boldsymbol{y}(0)|}\right) \quad . \qquad \textbf{(A5)}$$

$$= \lim_{t\to\infty} \frac{1}{t} \ln\left(\left|\boldsymbol{J}(\boldsymbol{x}) \cdot \frac{\boldsymbol{y}}{|\boldsymbol{y}(0)|}\right|\right)$$

For an n-dimensional system, there will be at most n distinct Lyapunov exponents for a given $\boldsymbol{x}(0)$, and the relevant exponent is picked out by the initial orientation $\boldsymbol{y}(0)/|\boldsymbol{y}(0)|$. The infinite time limit plays an important role in this analysis as it indicates that the Lyapunov exponents represent time-averaged quantities (meaning that transient behavior has decayed). The existence of this limit is guaranteed by Oseledec's (1969) multiplicative ergodic theorem, which holds under mild conditions. In addition, $\boldsymbol{J}$ is a constant in space in this limit (otherwise its value varies in space), and the Lyapunov exponents obtained from (A5) are then the same for almost every value of $\boldsymbol{x}(0)$. Hence, one often drops the dependence on the initial condition in (A5). Such exponents are usually called *global* Lyapunov exponents.

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